DETERMINANTS OF PERFECT COMPLEXES AND EULER CHARACTERISTICS IN RELATIVE K_0-GROUPS

MANUEL BREUNING

ABSTRACT. We study the K_0 and K_1-groups of exact and triangulated categories of perfect complexes, and we apply the results to show how determinant functors on triangulated categories can be used for the construction of Euler characteristics in relative algebraic K_0-groups.

1. INTRODUCTION

Let R be a ring and P a cochain complex of finitely generated projective R-modules. Then the Euler characteristic of P is defined to be the element $\chi(P) = \sum_i (-1)^i [P^i]$ in the Grothendieck group $K_0(R)$. The definition of this Euler characteristic can easily be generalized to perfect complexes of R-modules, i.e. complexes which in the derived category become isomorphic to bounded complexes of finitely generated projective R-modules.

Now suppose that we have a ring homomorphism $R \to S$. Recall that then there exists an exact sequence of K-groups

$$K_1(R) \longrightarrow K_1(S) \longrightarrow K_0(R, S) \longrightarrow K_0(R) \longrightarrow K_0(S),$$

where $K_0(R, S)$ is Swan's relative K_0-group. Let P be a perfect complex of R-modules for which $\chi(P)$ lies in the kernel of $K_0(R) \to K_0(S)$. In this situation, can we find a canonical preimage of $\chi(P)$ in $K_0(R, S)$? In certain cases this is indeed possible, provided the complex $S \otimes_R P$ is endowed with some natural additional data, and in recent years such refined Euler characteristics in $K_0(R, S)$ have found applications in arithmetic algebraic geometry. Sometimes they can be constructed using only elementary methods (see [3]), however a more general and conceptual approach is the use of determinant functors on exact categories of complexes (see [4], [2]).

In the applications, the relevant complexes often lie in derived categories and not in exact categories. Therefore it would be more natural to use determinant functors on triangulated categories (as defined in [1]) for the construction of Euler characteristics in $K_0(R, S)$. The purpose of this paper is to show that this is indeed possible.

The crucial observation is that the K-groups of a triangulated category of perfect complexes are closely related to classical K-groups. To state the precise result, we let R be a ring and denote the triangulated category of perfect complexes of R-modules by $\mathbf{D}^{\mathrm{perf}}(R)$. In [1] the K-groups $K_i\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$ for $i = 0$ and 1 were defined in terms of a universal determinant functor on $\mathbf{D}^{\mathrm{perf}}(R)$. These groups are related to the usual K-groups of R as follows.

Theorem 1.1. *There exist canonical homomorphisms*

$$K_i(R) \longrightarrow K_i\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$$

for $i = 0$ and 1. It is an isomorphism for $i = 0$ and surjective for $i = 1$. If R is regular then it is an isomorphism for $i = 1$.

The proof of this result involves studying the relation between determinant functors on the exact category of finitely generated projective R-modules, the exact category of perfect complexes of R-modules, and the triangulated category of perfect complexes of R-modules. Using this theorem it is then not difficult to show that in the construction of Euler characteristics in a relative K_0-group the determinant functors on exact categories can be replaced by determinant functors on triangulated categories.

This paper is structured as follows. We study determinant functors on exact and triangulated categories of perfect complexes in §2 and §3 respectively. In §5 we recall the homotopy fibre of a monoidal functor of Picard categories. Finally in §6 we explain the construction of Euler characteristics using determinant functors on triangulated categories.

In §4 we will give an example of a non-regular ring R for which the canonical surjection $K_1(R) \to K_1\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$ from Theorem 1.1 is not injective. Furthermore using the same example we will show that the isomorphism $K_1(\mathcal{A}) \cong K_1\big(\mathbf{D}^{\mathrm{b}}(\mathcal{A})\big)$ which was proved in [1, §5] for abelian categories $\mathcal{A}$ does not generalize to exact categories. The results from §4 are not used in the subsequent sections.

We refer the reader to [1] for the definition and properties of determinant functors on exact and triangulated categories. Moreover we use the same notations as in [1], in particular we recall that if $\mathcal{P}$ is a Picard category, $\mathcal{E}$ is an exact category and w a class of morphism in $\mathcal{E}$ which contains all isomorphisms and is closed under composition then $\det((\mathcal{E}, w), \mathcal{P})$ denotes the category whose objects are the determinant functors $f = (f_1, f_2) : (\mathcal{E}, w) \to \mathcal{P}$, and we omit w from the notation if $w = \mathrm{iso}$ is the class of all isomorphisms.

2. Determinant functors on exact categories of perfect complexes

Let R be a ring (associative with unit). Unless otherwise stated all R-modules will be left R-modules, and all complexes of R-modules will be cochain complexes of left R-modules. The abelian category of all complexes of R-modules will be denoted by $\mathbf{C}(R)$. If M is an R-module then $M[0]$ denotes the complex which consists of the module M in degree 0 and of zero modules in all other degrees.

Let $\mathcal{E}$ be a full exact subcategory of $\mathbf{C}(R)$. A morphism in $\mathcal{E}$ will be called a quasi-isomorphism if it becomes a quasi-isomorphism after embedding in $\mathbf{C}(R)$, i.e. if it induces an isomorphism on cohomology. The class of all quasi-isomorphisms in $\mathcal{E}$ will be denoted by qis. In the following we will frequently use the following lemma (which is proved in the same way as [5, Corollary 2.12]).

Lemma 2.1. *Assume that $\mathcal{E}$ is a full exact subcategory of $\mathbf{C}(R)$ which is closed under mapping cones (i.e. if a is any morphism in $\mathcal{E}$ then $\mathrm{cone}(a)$ is an object of $\mathcal{E}$). Let $f = (f_1, f_2) : (\mathcal{E}, \mathrm{qis}) \to \mathcal{P}$ be a determinant functor. Then for any homotopic quasi-isomorphisms a and b in $\mathcal{E}$ we have $f_1(a) = f_1(b)$ in $\mathcal{P}$.*

We remark that the assumptions of the lemma are satisfied for the subcategories $\mathbf{C}^{\mathrm{b}}(R\text{-proj})$, $\mathbf{C}^{\mathrm{b}}(R\text{-mod})$ and $\mathbf{C}^{\mathrm{perf}}(R)$ which are considered later in this paper.

Let R-proj denote the category of finitely generated projective R-modules and $\mathbf{C}^{\mathrm{b}}(R\text{-proj})$ the category of bounded complexes of objects of R-proj. The categories R-proj and $\mathbf{C}^{\mathrm{b}}(R\text{-proj})$ are both exact and the embedding $R\text{-proj} \to \mathbf{C}^{\mathrm{b}}(R\text{-proj})$, $M \mapsto M[0]$, is an exact functor. For every Picard category $\mathcal{P}$ we obtain an induced functor

$$(1) \qquad \det\big((\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}), \mathcal{P}\big) \longrightarrow \det(R\text{-proj}, \mathcal{P})$$

which by [5, Theorem 2.3] is an equivalence of categories.

Definition 2.2. A complex A of R-modules is said to be *perfect* if there exists a bounded complex of finitely generated projective R-modules P and a quasi-isomorphism $P \to A$.

The full subcategory of $\mathbf{C}(R)$ consisting of all perfect complexes will be denoted by $\mathbf{C}^{\mathrm{perf}}(R)$. Note that $\mathbf{C}^{\mathrm{perf}}(R)$ is an exact category and that the embedding $\mathbf{C}^{\mathrm{b}}(R\text{-proj}) \to \mathbf{C}^{\mathrm{perf}}(R)$ is an exact functor. Hence for every Picard category $\mathcal{P}$ we obtain an induced functor

$$(2) \qquad \det\big((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\big) \longrightarrow \det\big((\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}), \mathcal{P}\big).$$

Lemma 2.3. *The functor (2) is an equivalence of categories.*

Remark 2.4. Before sketching the proof of this lemma we recall a useful property of objects in $\mathbf{C}^{\mathrm{b}}(R\text{-proj})$ which will be used repeatedly in this paper. Let $s : A \to B$ be a quasi-isomorphism in $\mathbf{C}(R)$ and let $b : P \to B$ be any morphism of complexes where P is an object in $\mathbf{C}^{\mathrm{b}}(R\text{-proj})$ (or, more generally, P is a bounded above complex of projective R-modules). Then there exists a morphism of complexes $a : P \to A$ such that the diagram

$$
\begin{array}{ccc}
 & & A \\
 & {\scriptstyle a}\nearrow & \Big\downarrow {\scriptstyle s} \\
P & \xrightarrow{\ b\ } & B
\end{array}
$$

commutes up to homotopy. Furthermore this morphism a is unique up to homotopy.

Proof of Lemma 2.3. This is essentially known (see e.g. [6, Theorem 2]), so we only sketch the main idea. For every perfect complex C we fix a quasi-isomorphism $q_C : P_C \to C$ where P_C is in $\mathbf{C}^{\mathrm{b}}(R\text{-proj})$. Then we can construct a functor

$$(3) \qquad \det\big((\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}), \mathcal{P}\big) \longrightarrow \det\big((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\big)$$

as follows. Given a determinant functor $f = (f_1, f_2) : (\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}) \to \mathcal{P}$ we define $g_1 : \mathbf{C}^{\mathrm{perf}}(R)_{\mathrm{qis}} \to \mathcal{P}$ by $g_1(C) := f_1(P_C)$ and for a quasi-isomorphism $a : C \to D$ by $g_1(a) := f_1(b)$ where $b : P_C \to P_D$ is a quasi-isomorphism such that $q_D \circ b$ is homotopic to $a \circ q_C$ (this is well-defined because such a map b is unique up to homotopy). If $\Delta : 0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathbf{C}^{\mathrm{perf}}(R)$ then there exists a short exact sequence $\Delta' : 0 \to A' \to B' \to C' \to 0$ in $\mathbf{C}^{\mathrm{b}}(R\text{-proj})$ and a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$

where the vertical maps are quasi-isomorphisms. Furthermore there are quasi-isomorphisms $u : P_A \to A'$, $v : P_B \to B'$ and $w : P_C \to C'$ such that $a \circ u$, $b \circ v$ and $c \circ w$ are homotopic to q_A, q_B and q_C respectively. We define $g_2(\Delta) : g_1(B) \to g_1(A) \otimes g_1(C)$ to be $g_2(\Delta) := (f_1(u) \otimes f_1(w))^{-1} \circ f_2(\Delta') \circ f_1(v)$. One can verify that $g_2(\Delta)$ is well-defined and that (g_1, g_2) is a determinant functor in $\det\big((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\big)$.

It is obvious how to define the functor (3) on morphisms of determinant functors. One then easily checks that this functor (3) is a quasi-inverse of the functor (2). $\square$

Corollary 2.5. *For every Picard category $\mathcal{P}$ the canonical functor*

$$\det\big((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\big) \to \det(R\text{-proj}, \mathcal{P})$$

which is induced by the embedding R-proj $\to \mathbf{C}^{\mathrm{perf}}(R)$ is an equivalence of categories. Hence the canonical map

$$K_i(R) \longrightarrow K_i\big(\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}\big)$$

is an isomorphism.

Proof. The first statement is immediate by composing the equivalences (1) and (2). The second statement follows from the first statement and the relation between K-groups and universal determinant functors, see Lemma 2.6 below. $\square$

Lemma 2.6. *Let $\mathcal{E}$ and $\mathcal{E}'$ be exact category. Let w and w' be classes of morphisms in $\mathcal{E}$ and $\mathcal{E}'$ respectively which contain all isomorphisms and are closed under composition. Let $F : \mathcal{E} \to \mathcal{E}'$ be an exact functor with $F(w) \subseteq w'$. Then for every Picard category $\mathcal{P}$ there is an induced functor*

$$(4) \qquad\qquad \det((\mathcal{E}', w'), \mathcal{P}) \longrightarrow \det((\mathcal{E}, w), \mathcal{P}),$$

and there are induced homomorphisms

$$(5) \qquad\qquad K_i(\mathcal{E}, w) \longrightarrow K_i(\mathcal{E}', w')$$

for $i = 0, 1$. The following are equivalent.

(i) *For every Picard category $\mathcal{P}$ the functor (4) is an equivalence of categories.*
(ii) *The homomorphisms (5) are isomorphisms for $i = 0$ and 1.*

Proof. Let $f : (\mathcal{E}, w) \to \mathcal{V}$ and $f' : (\mathcal{E}', w') \to \mathcal{V}'$ be universal determinant functors, and let $M : \mathcal{V} \to \mathcal{V}'$ be a monoidal functor such that $M \circ f \cong f' \circ F$ (by the definition of a universal determinant functor such an M exists and is unique up to isomorphism). Then there exists the following diagram of categories and functors

$$
\begin{array}{ccc}
\det((\mathcal{E}', w'), \mathcal{P}) & \xrightarrow{\;g \mapsto g \circ F\;} & \det((\mathcal{E}, w), \mathcal{P}) \\
\uparrow & & \uparrow \\
\mathrm{Hom}^{\otimes}(\mathcal{V}', \mathcal{P}) & \xrightarrow{\;N \mapsto N \circ M\;} & \mathrm{Hom}^{\otimes}(\mathcal{V}, \mathcal{P})
\end{array}
$$

which commutes up to natural isomorphism. Since the vertical functors are equivalences, it follows that the top horizontal functor is an equivalence if and only if the bottom horizontal functor is an equivalence. Clearly the top horizontal functor is the functor (4). On the other hand it is easy to see that the bottom horizontal functor is an equivalence for all $\mathcal{P}$ if and only if the monoidal functor $M : \mathcal{V} \to \mathcal{V}'$ is an equivalence of Picard categories, and by [1, Lemma 2.2] this is the case if and

only if $K_i(\mathcal{E}, w) = \pi_i(\mathcal{V}) \xrightarrow{\pi_i(M)} \pi_i(\mathcal{V}') = K_i(\mathcal{E}', w')$ is an isomorphism for $i = 0$ and $i = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We remark that a statement similar to Lemma 2.6 is also valid for exact functors $F : \mathcal{T} \to \mathcal{T}'$ of triangulated categories $\mathcal{T}, \mathcal{T}'$ and for certain functors $F : \mathcal{E} \to \mathcal{T}$ from an exact category $\mathcal{E}$ to a triangulated category $\mathcal{T}$.

3. Determinant functors on triangulated categories of perfect complexes

Let R be a ring. The derived category of the abelian category of all R-modules will be denoted by $\mathbf{D}(R)$. Thus the objects in $\mathbf{D}(R)$ are the complexes of R-modules, and a morphism $A \to B$ in $\mathbf{D}(R)$ is an equivalence class of diagrams $A \xleftarrow{s} C \xrightarrow{a} B$ where a is any morphism of complexes and s is a quasi-isomorphism. The category $\mathbf{D}(R)$ is triangulated, where as in [2, bottom of p. 28] we choose the triangulation in which a triangle is distinguished if it is isomorphic to a triangle of the form $A \xrightarrow{a} B \to \mathrm{cone}(a) \to A[1]$, where $B \to \mathrm{cone}(a)$ is the canonical inclusion and $\mathrm{cone}(a) \to A[1]$ is the negative of the canonical projection. The full subcategory of $\mathbf{D}(R)$ consisting of all perfect complexes will be denoted by $\mathbf{D}^{\mathrm{perf}}(R)$. One easily verifies that $\mathbf{D}^{\mathrm{perf}}(R)$ is a triangulated subcategory of $\mathbf{D}(R)$. We let $I : \mathbf{C}^{\mathrm{perf}}(R) \to \mathbf{D}^{\mathrm{perf}}(R)$ denote the canonical functor.

Lemma 3.1. *The functor $I : \mathbf{C}^{\mathrm{perf}}(R) \to \mathbf{D}^{\mathrm{perf}}(R)$ induces a functor*

$$\det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right) \longrightarrow \det\left((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\right),$$
$$g \longmapsto g \circ I,$$

for every Picard category $\mathcal{P}$, and it therefore induces homomorphisms

$$K_i\left(\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}\right) \longrightarrow K_i\left(\mathbf{D}^{\mathrm{perf}}(R)\right)$$

for $i = 0, 1$.

Proof. We first recall that for every short exact sequence $\Delta : 0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ of complexes of R-modules there exists an associated distinguished triangle $\widehat{\Delta} : A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$ in $\mathbf{D}(R)$ where c is the morphism $C \xleftarrow{s} \mathrm{cone}(a) \to TA$ (here $s : \mathrm{cone}(a) \to C$ is the canonical quasi-isomorphism and $\mathrm{cone}(a) \to TA$ is the negative of the canonical projection).

Let $g = (g_1, g_2) : \mathbf{D}^{\mathrm{perf}}(R) \to \mathcal{P}$ be a determinant functor. The functor $I : \mathbf{C}^{\mathrm{perf}}(R) \to \mathbf{D}^{\mathrm{perf}}(R)$ sends quasi-isomorphisms in $\mathbf{C}^{\mathrm{perf}}(R)$ to isomorphisms in $\mathbf{D}^{\mathrm{perf}}(R)$. We can therefore define $f_1 := g_1 \circ I : \mathbf{C}^{\mathrm{perf}}(R)_{\mathrm{qis}} \to \mathcal{P}$. For a short exact sequence $\Delta : 0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ in $\mathbf{C}^{\mathrm{perf}}(R)$ we define $f_2(\Delta) := g_2(\widehat{\Delta}) : f_1(B) \to f_1(A) \otimes f_1(C)$. It is not difficult to verify that (f_1, f_2) is a determinant functor on $(\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis})$ which we will denote by $g \circ I$. For a morphism $\lambda : g \to g'$ of determinant functors $g, g' : \mathbf{D}^{\mathrm{perf}}(R) \to \mathcal{P}$ it is clear how to define the morphism $\lambda \circ I : g \circ I \to g' \circ I$. We obtain a functor

$$\det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right) \longrightarrow \det\left((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\right).$$

Now if $f : (\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}) \to \mathcal{V}$ and $g : \mathbf{D}^{\mathrm{perf}}(R) \to \mathcal{W}$ are universal determinant functors then there exists a monoidal functor $M : \mathcal{V} \to \mathcal{W}$ such that $M \circ f$ and

$g \circ I$ are isomorphic. Furthermore M is unique up to isomorphism. Therefore M induces well-defined group homomorphisms

$$K_i\big(\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}\big) = \pi_i(\mathcal{V}) \xrightarrow{\pi_i(M)} \pi_i(\mathcal{W}) = K_i\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$$

for $i = 0, 1$. $\hfill\square$

We will now study the maps $K_i\big(\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}\big) \to K_i\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$ in more detail, first for arbitrary rings (Proposition 3.2) and then for regular rings (Proposition 3.4).

Proposition 3.2. *Let R be an arbitrary ring. Then the homomorphism*

$$K_i\big(\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}\big) \longrightarrow K_i\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$$

is bijective for $i = 0$ and surjective for $i = 1$.

Proof. Let $f = (f_1, f_2) : (\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}) \to \mathcal{V}$ be a universal determinant functor. We will construct a universal determinant functor for the triangulated category $\mathbf{D}^{\mathrm{perf}}(R)$ by identifying certain morphisms in the Picard category $\mathcal{V}$.

We first note that the functor $f_1 : \mathbf{C}^{\mathrm{perf}}(R)_{\mathrm{qis}} \to \mathcal{V}$ naturally induces a functor $\tilde{f}_1 : \mathbf{D}^{\mathrm{perf}}(R)_{\mathrm{iso}} \to \mathcal{V}$. Indeed, if A is an object in $\mathbf{D}^{\mathrm{perf}}(R)$ then we let $\tilde{f}_1(A) := f_1(A)$, and if $a : A \to B$ is an isomorphism in $\mathbf{D}^{\mathrm{perf}}(R)$ then we let $\tilde{f}_1(a) := f_1(t) \circ f_1(s)^{-1}$ for any quasi-isomorphisms $s : C \to A, t : C \to B$ in $\mathbf{C}^{\mathrm{perf}}(R)$ such that $a = t \circ s^{-1}$ in $\mathbf{D}^{\mathrm{perf}}(R)$. Note that $f_1 = \tilde{f}_1 \circ I$.

Now let $S' \subseteq \bigcup_{(X,Y)} \mathrm{Hom}_{\mathcal{V}}(X, Y) \times \mathrm{Hom}_{\mathcal{V}}(X, Y)$ (where the union is over all pairs (X, Y) of objects of $\mathcal{V}$) be the class consisting of the following pairs of morphisms in $\mathcal{V}$. If $\Delta_i : 0 \to A_i \to B_i \to C_i \to 0$ are short exact sequences in $\mathbf{C}^{\mathrm{perf}}(R)$ for $i = 1, 2$ and

$$
\begin{array}{ccccccc}
\widehat{\Delta_1} : & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & TA_1 \\
& \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle Ta} \\
\widehat{\Delta_2} : & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & TA_2
\end{array}
$$

is a commutative diagram in $\mathbf{D}^{\mathrm{perf}}(R)$ with isomorphisms a, b, c in $\mathbf{D}^{\mathrm{perf}}(R)$, then the pair

$$\big((\tilde{f}_1(a) \otimes \tilde{f}_1(c)) \circ f_2(\Delta_1), \quad f_2(\Delta_2) \circ \tilde{f}_1(b)\big)$$

belongs to S'.

Let $Q' : \mathcal{V} \to \mathcal{V}/S'$ be the quotient Picard category (cf. Lemma 3.3 below), and let $\overline{f}_1$ be the composite functor $\mathbf{D}^{\mathrm{perf}}(R)_{\mathrm{iso}} \xrightarrow{\tilde{f}_1} \mathcal{V} \xrightarrow{Q'} \mathcal{V}/S'$. If $\Delta : A \to B \to C \to TA$ is a distinguished triangle in $\mathbf{D}^{\mathrm{perf}}(R)$ then there exists a short exact sequence $\Delta_1 : 0 \to A_1 \to B_1 \to C_1 \to 0$ in $\mathbf{C}^{\mathrm{perf}}(R)$ and isomorphisms a, b, c in $\mathbf{D}^{\mathrm{perf}}(R)$ such that
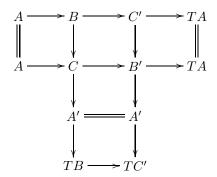
$$
\begin{array}{ccccccc}
\Delta : & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\
& \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle Ta} \\
\widehat{\Delta_1} : & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & TA_1
\end{array}
$$

commutes in $\mathbf{D}^{\mathrm{perf}}(R)$. We define $\overline{f}_2(\Delta) : \overline{f}_1(B) \to \overline{f}_1(A) \otimes \overline{f}_1(C)$ to be $\overline{f}_2(\Delta) := (\overline{f}_1(a^{-1}) \otimes \overline{f}_1(c^{-1})) \circ f_2(\Delta_1) \circ \overline{f}_1(b)$. Our definition of S' guarantees that $\overline{f}_2(\Delta)$ is well-defined.

Now $\overline{f} = (\overline{f}_1, \overline{f}_2) : \mathbf{D}^{\mathrm{perf}}(R) \to \mathcal{V}/S'$ does not necessarily satisfy the associativity axiom. Let S'' be the set of pairs of morphisms in $\mathcal{V}/S'$ which must be identified in order for the associativity axiom to hold. More precisely, if

$$
\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & C' & \longrightarrow & TA \\
\| & & \downarrow & & \downarrow & & \| \\
A & \longrightarrow & C & \longrightarrow & B' & \longrightarrow & TA \\
& & \downarrow & & \downarrow & & \\
& & A' & = & A' & & \\
& & \downarrow & & \downarrow & & \\
& & TB & \longrightarrow & TC' & &
\end{array}
$$

is an octahedral diagram in $\mathbf{D}^{\mathrm{perf}}(R)$, then the pair

$$
\left(\varphi \circ (\mathrm{id} \otimes \overline{f}_2(\Delta_{\mathrm{v}2})) \circ \overline{f}_2(\Delta_{\mathrm{h}2}), \quad (\overline{f}_2(\Delta_{\mathrm{h}1}) \otimes \mathrm{id}) \circ \overline{f}_2(\Delta_{\mathrm{v}1}) \right)
$$

belongs to S''. Here $\Delta_{\mathrm{h}1}$ and $\Delta_{\mathrm{h}2}$ (resp. $\Delta_{\mathrm{v}1}$ and $\Delta_{\mathrm{v}2}$) denote the first and second horizontal (resp. vertical) distinguished triangles in the octahedral diagram, and φ is the associativity constraint in the Picard category $\mathcal{V}/S'$.

Let $Q'' : \mathcal{V}/S' \to (\mathcal{V}/S')/S'' =: \mathcal{W}$ be the quotient Picard category (cf. Lemma 3.3). Let $g_1 := Q'' \circ \overline{f}_1 : \mathbf{D}^{\mathrm{perf}}(R)_{\mathrm{iso}} \to \mathcal{W}$ and for every distinguished triangle $\Delta : A \to B \to C \to TA$ in $\mathbf{D}^{\mathrm{perf}}(R)$ let $g_2(\Delta)$ be the map $g_1(B) = Q''(\overline{f}_1(B)) \xrightarrow{Q''(\overline{f}_2(\Delta))} Q''(\overline{f}_1(A) \otimes \overline{f}_1(C)) = g_1(A) \otimes g_1(C)$. It is then easy to check that $g = (g_1, g_2) : \mathbf{D}^{\mathrm{perf}}(R) \to \mathcal{W}$ is a determinant functor. Furthermore $Q \circ f = g \circ I$ where $Q := Q'' \circ Q' : \mathcal{V} \to \mathcal{W}$.

We claim that $g : \mathbf{D}^{\mathrm{perf}}(R) \to \mathcal{W}$ is universal. For this we must show that for every Picard category $\mathcal{P}$ the functor

(6)
$$
\begin{aligned}
\mathrm{Hom}^{\otimes}(\mathcal{W}, \mathcal{P}) &\longrightarrow \det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right), \\
M &\longmapsto M \circ g,
\end{aligned}
$$

is an equivalence of categories.

Let $h : \mathbf{D}^{\mathrm{perf}}(R) \to \mathcal{P}$ be any determinant functor. Then there exists a monoidal functor $N : \mathcal{V} \to \mathcal{P}$ such that the determinant functors $N \circ f$ and $h \circ I$ are isomorphic. One easily sees that N factors as $\mathcal{V} \xrightarrow{Q} \mathcal{W} \xrightarrow{\overline{N}} \mathcal{P}$ for a unique monoidal functor $\overline{N} : \mathcal{W} \to \mathcal{P}$. Thus the determinant functors $\overline{N} \circ g \circ I$ and $h \circ I$ are isomorphic in $\det\left((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\right)$, and this implies that the determinant functors $\overline{N} \circ g$ and h are isomorphic in $\det(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P})$. Hence the functor (6) is essentially surjective.

To show that (6) is fully faithful, we consider the following commutative diagram.

$$
\begin{array}{ccc}
\mathrm{Hom}^{\otimes}(\mathcal{W}, \mathcal{P}) & \xrightarrow{\ M \mapsto M \circ g\ } & \det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right) \\
{\scriptstyle M \mapsto M \circ Q} \downarrow & & \downarrow {\scriptstyle h \mapsto h \circ I} \\
\mathrm{Hom}^{\otimes}(\mathcal{V}, \mathcal{P}) & \xrightarrow{\ N \mapsto N \circ f\ } & \det\left((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\right)
\end{array}
$$

The bottom horizontal functor is fully faithful since $f : (\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}) \to \mathcal{V}$ is a universal determinant functor. Furthermore it is easy to see that the left vertical

functor is fully faithful and that the right vertical functor is faithful. It follows that the top horizontal functor is fully faithful as required. This finishes the proof of the claim that g is universal.

By Lemma 3.3 the homomorphisms $\pi_i(Q')$ and $\pi_i(Q'')$ are bijective for $i = 0$ and surjective for $i = 1$. Therefore the same is true for $\pi_i(Q) = \pi_i(Q'') \circ \pi_i(Q')$. It follows that the homomorphism

$$K_i\big(\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}\big) = \pi_i(\mathcal{V}) \xrightarrow{\pi_i(Q)} \pi_i(\mathcal{W}) = K_i\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$$

is bijective for $i = 0$ and surjective for $i = 1$. $\qquad\square$

The following lemma describes the quotient Picard category which was used in the proof of Proposition 3.2.

Lemma 3.3. *Let $\mathcal{V}$ be a Picard category and assume that we are given a class $S \subseteq \bigcup_{(X,Y)} \big(\mathrm{Hom}_{\mathcal{V}}(X,Y) \times \mathrm{Hom}_{\mathcal{V}}(X,Y)\big)$ where the union is over all pairs (X,Y) of objects in $\mathcal{V}$. Then there exists a Picard category $\mathcal{V}/S$ and a monoidal functor $Q : \mathcal{V} \to \mathcal{V}/S$ such that the following two properties are satisfied.*

(1) $Q(\alpha) = Q(\alpha')$ whenever $(\alpha, \alpha') \in S$
(2) If $\mathcal{P}$ is any Picard category and $M : \mathcal{V} \to \mathcal{P}$ a monoidal functor such that $M(\alpha) = M(\alpha')$ in $\mathcal{P}$ whenever $(\alpha, \alpha') \in S$ then there exists a unique monoidal functor $N : \mathcal{V}/S \to \mathcal{P}$ making the diagram

$$
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\ Q\ } & \mathcal{V}/S \\
{\scriptstyle M}\big\downarrow & \swarrow{\scriptstyle N} & \\
\mathcal{P} & &
\end{array}
$$

commutative.

If $\mathcal{V}$ is small then the functor $Q : \mathcal{V} \to \mathcal{V}/S$ induces an isomorphism $\pi_0(Q) : \pi_0(\mathcal{V}) \cong \pi_0(\mathcal{V}/S)$ and a surjection $\pi_1(Q) : \pi_1(\mathcal{V}) \to \pi_1(\mathcal{V}/S)$. If S contains a pair (α, α') with $\alpha \neq \alpha'$ then the surjection $\pi_1(Q) : \pi_1(\mathcal{V}) \to \pi_1(\mathcal{V}/S)$ is not an isomorphism.

Proof. Suppose that for every pair (X,Y) of objects in $\mathcal{V}$ we have an equivalence relation $\sim_{(X,Y)}$ on $\mathrm{Hom}_{\mathcal{V}}(X,Y)$. Then we say that these equivalence relations form a compatible system if they satisfy the following three conditions:

(1) If $\alpha, \alpha' : X \to Y$ are morphisms such that $(\alpha, \alpha') \in S$ then $\alpha \sim_{(X,Y)} \alpha'$.
(2) If $\alpha \sim_{(X,Y)} \alpha'$ and $\beta \sim_{(Y,Z)} \beta'$ then $\beta \circ \alpha \sim_{(X,Z)} \beta' \circ \alpha'$.
(3) If $\alpha \sim_{(X,Y)} \alpha'$ and $\beta \sim_{(Z,W)} \beta'$ then $\alpha \otimes \beta \sim_{(X \otimes Z, Y \otimes W)} \alpha' \otimes \beta'$.

Let $\{\sim_{(X,Y)} : (X,Y) \in \mathrm{obj}(\mathcal{V})^2\}$ be the unique minimal compatible system of equivalence relations. We define $\mathcal{V}/S$ to be the category with objects $\mathrm{obj}(\mathcal{V}/S) := \mathrm{obj}(\mathcal{V})$ and morphisms $\mathrm{Hom}_{\mathcal{V}/S}(X,Y) := \mathrm{Hom}_{\mathcal{V}}(X,Y)/\sim_{(X,Y)}$. Then $\mathcal{V}/S$ is a Picard category in a natural way and there exists a canonical monoidal functor $Q : \mathcal{V} \to \mathcal{V}/S$. Furthermore it is easy to verify the universal property for $\mathcal{V}/S$.

By construction, the functor $Q : \mathcal{V} \to \mathcal{V}/S$ is bijective on objects and surjective on morphisms. Hence the induced homomorphisms $\pi_i(Q) : \pi_i(\mathcal{V}) \to \pi_i(\mathcal{V}/S)$ are certainly surjective. The map $\pi_0(Q)$ is injective because any isomorphism in $\mathcal{V}/S$ lifts to an isomorphism in $\mathcal{V}$. Finally, if $\alpha, \alpha' : X \to Y$ are two distinct morphisms in $\mathcal{V}$ with $(\alpha, \alpha') \in S$, then the element $\alpha^{-1} \circ \alpha' \in \mathrm{Aut}_{\mathcal{V}}(X) = \pi_1(\mathcal{V})$ is non-trivial but becomes trivial in $\pi_1(\mathcal{V}/S)$. $\qquad\square$

Recall that a ring R is called regular if R is noetherian and every R-module has a finite projective resolution.

Proposition 3.4. *If the ring R is regular, then for every Picard category $\mathcal{P}$ the functor* $\det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right) \to \det\left((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\right)$ *induced by* $I : \mathbf{C}^{\mathrm{perf}}(R) \to \mathbf{D}^{\mathrm{perf}}(R)$ *is an equivalence of categories. Hence in this case the homomorphism*

$$K_i\left(\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}\right) \longrightarrow K_i\left(\mathbf{D}^{\mathrm{perf}}(R)\right)$$

is an isomorphism for $i = 0$ and $i = 1$.

Proof. Let $F : \det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right) \to \det(R\text{-proj}, \mathcal{P})$ be the functor which is induced by the embedding $R\text{-proj} \to \mathbf{D}^{\mathrm{perf}}(R)$. We claim that F is an equivalence of categories.

Let R-mod denote the abelian category of finitely generated R-modules and $\mathbf{D}^{\mathrm{b}}(R\text{-mod})$ the bounded derived category of R-mod. Since R is regular every bounded complex of finitely generated R-modules is perfect. On the other hand, every perfect complex of R-modules is isomorphic in $\mathbf{D}^{\mathrm{perf}}(R)$ to a bounded complex of finitely generated projective modules and so in particular to a bounded complex of finitely generated modules. It easily follows that there exists a canonical equivalence of triangulated categories $\mathbf{D}^{\mathrm{b}}(R\text{-mod}) \to \mathbf{D}^{\mathrm{perf}}(R)$.

Now the functor $F : \det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right) \to \det(R\text{-proj}, \mathcal{P})$ can be factored as

$$\det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right) \xrightarrow{(1)} \det\left(\mathbf{D}^{\mathrm{b}}(R\text{-mod}), \mathcal{P}\right)$$

$$\xrightarrow{(2)} \det(R\text{-mod}, \mathcal{P})$$

$$\xrightarrow{(3)} \det(R\text{-proj}, \mathcal{P}).$$

Here the functor (1) is induced by the equivalence of triangulated categories $\mathbf{D}^{\mathrm{b}}(R\text{-mod}) \to \mathbf{D}^{\mathrm{perf}}(R)$ and is therefore itself an equivalence. The functor (2) is induced by the canonical functor $R\text{-mod} \to \mathbf{D}^{\mathrm{b}}(R\text{-mod})$ and is an equivalence by the theorem of the heart [1, Theorem 5.2]. Finally the functor (3) is induced by the inclusion $R\text{-proj} \to R\text{-mod}$. By Quillen's resolution theorem [8, Corollary 2 in §4] this inclusion induces an isomorphism $K_i(R\text{-proj}) \cong K_i(R\text{-mod})$ for all i, hence by Lemma 2.6 the functor (3) is an equivalence. It follows that F is an equivalence as claimed.

Now note that the equivalence F can also be factored as

$$\det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right) \longrightarrow \det\left((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\right) \longrightarrow \det(R\text{-proj}, \mathcal{P}).$$

Since the functor $\det\left((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\right) \to \det(R\text{-proj}, \mathcal{P})$ is an equivalence by Corollary 2.5, it follows that $\det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right) \to \det\left((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\right)$ is an equivalence. This proves the first statement of the proposition.

We have shown that for every Picard category $\mathcal{P}$ the functor $\det\left(\mathbf{D}^{\mathrm{perf}}(R), \mathcal{P}\right) \to \det\left((\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}), \mathcal{P}\right)$ is an equivalence of categories. By an argument similar to Lemma 2.6 this implies that $K_i\left(\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}\right) \to K_i\left(\mathbf{D}^{\mathrm{perf}}(R)\right)$ is an isomorphism for $i = 0$ and $i = 1$. $\qquad\square$

We can now prove Theorem 1.1 from the introduction.

Proof of Theorem 1.1. The composite of the canonical homomorphisms

$$K_i(R) \longrightarrow K_i\left(\mathbf{C}^{\mathrm{perf}}(R), \mathrm{qis}\right) \longrightarrow K_i\left(\mathbf{D}^{\mathrm{perf}}(R)\right)$$

from Corollary 2.5 and Lemma 3.1 gives a canonical homomorphism $K_i(R) \to K_i\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$. The statements about bijectivity and surjectivity follow from Corollary 2.5, Proposition 3.2 and Proposition 3.4. $\hfill\square$

4. An example

We have seen that for a regular ring R the canonical map $K_1(R) \to K_1\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$ is an isomorphism. In this section we will give an example of a non-regular ring R for which the groups $K_1(R)$ and $K_1\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$ are not isomorphic. The same example also shows that in general the groups $K_1(R\text{-proj})$ and $K_1\big(\mathbf{D}^{\mathrm{b}}(R\text{-proj})\big)$ are not isomorphic, so the isomorphism $K_1(\mathcal{A}) \cong K_1\big(\mathbf{D}^{\mathrm{b}}(\mathcal{A})\big)$ for an abelian category $\mathcal{A}$ (compare [1, §5.1]) does not generalize to exact categories. The example in this section is motivated by [10, §2].

For any ring R we let $\mathbf{K}^{\mathrm{b}}(R\text{-proj})$ be the bounded homotopy category of R-proj, so the objects of $\mathbf{K}^{\mathrm{b}}(R\text{-proj})$ are bounded complexes of finitely generated projective R-modules and the morphisms are homotopy classes of morphisms of complexes. It is well known that $\mathbf{K}^{\mathrm{b}}(R\text{-proj})$ has the structure of a triangulated category.

There exists a canonical functor $\mathbf{C}^{\mathrm{b}}(R\text{-proj}) \to \mathbf{K}^{\mathrm{b}}(R\text{-proj})$. A quasi-isomorphism $a : A \to B$ in $\mathbf{C}^{\mathrm{b}}(R\text{-proj})$ is mapped to an isomorphism in $\mathbf{K}^{\mathrm{b}}(R\text{-proj})$. Hence, as in the proof of Lemma 3.1, if $\Delta : 0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ is a short exact sequence in $\mathbf{C}^{\mathrm{b}}(R\text{-proj})$, there exists a canonical morphism $c : C \to TA$ in $\mathbf{K}^{\mathrm{b}}(R\text{-proj})$ such that $\widehat{\Delta} : A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$ is a distinguished triangle in $\mathbf{K}^{\mathrm{b}}(R\text{-proj})$.

Lemma 4.1. *For any ring R the canonical functor $\mathbf{C}^{\mathrm{b}}(R\text{-proj}) \to \mathbf{K}^{\mathrm{b}}(R\text{-proj})$ induces a functor*

(7) $$\det\big(\mathbf{K}^{\mathrm{b}}(R\text{-proj}), \mathcal{P}\big) \longrightarrow \det\big((\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}), \mathcal{P}\big)$$

and therefore a homomorphism

(8) $$K_1\big(\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}\big) \longrightarrow K_1\big(\mathbf{K}^{\mathrm{b}}(R\text{-proj})\big).$$

The homomorphism (8) is always surjective. If $R = k[\varepsilon]/(\varepsilon^2)$ for a field k then the homomorphism (8) is not injective.

Proof. The proof of the existence of the functor (7) and homomorphism (8) is essentially the same as the proof of Lemma 3.1, and the proof of the surjectivity of (8) is similar to the proof of Proposition 3.2. More precisely, if $f = (f_1, f_2) : (\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}) \to \mathcal{V}$ is a universal determinant functor then we can construct a universal determinant functor $g = (g_1, g_2) : \mathbf{K}^{\mathrm{b}}(R\text{-proj}) \to \mathcal{W}$ where $\mathcal{W}$ is obtained from $\mathcal{V}$ by identifying certain homomorphisms. We denote the corresponding monoidal functor $\mathcal{V} \to \mathcal{W}$ by Q. It follows from Lemma 3.3 that (8) is surjective.

From now on let $R = k[\varepsilon]/(\varepsilon^2)$ for some field k. To show that the homomorphism (8) is not injective, it suffices to show that there exist two morphisms $\alpha, \alpha' : X \to Y$ in $\mathcal{V}$ such that $\alpha \neq \alpha'$ but $Q(\alpha) = Q(\alpha')$. We claim that the two morphisms $\alpha = \mathrm{id} : f_1(R[0]) \to f_1(R[0])$ and $\alpha' = f_1(1 + \varepsilon) : f_1(R[0]) \to f_1(R[0])$ (where the homomorphism $1 + \varepsilon : R[0] \to R[0]$ is given by multiplication with $1 + \varepsilon$) have these properties.

We first show that $\alpha \neq \alpha'$. Recall that $K_1(R)$ can be described in terms of generators and relations, where the generators are pairs (P, a) with $P \in \mathrm{obj}(R\text{-proj})$ and $a : P \to P$ an automorphism. Since R is a commutative local ring, the usual determinant gives an isomorphism $K_1(R) \cong R^\times$. It follows that (R, id)

and $(R, 1 + \varepsilon)$ are distinct in $K_1(R)$. Since $K_1(R) \cong K_1(R\text{-proj})$ we can deduce that $h_1(\mathrm{id})$ and $h_1(1 + \varepsilon)$ are distinct in $K_1(R\text{-proj})$ where (h_1, h_2) is a universal determinant functor on R-proj. Finally this implies that $\alpha = f_1(\mathrm{id})$ and $\alpha' = f_1(1 + \varepsilon)$ are distinct in $K_1\big(\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}\big)$ because the canonical map $K_1(R\text{-proj}) \to K_1\big(\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}\big)$ is an isomorphism.

Next we show that $Q(\alpha) = Q(\alpha')$. For this it suffices to show that $g_1(\mathrm{id}) = g_1(1 + \varepsilon)$. However this follows immediately from the commutative diagram in $\mathbf{K}^{\mathrm{b}}(R\text{-proj})$

$$
\begin{array}{ccccccc}
R[0] & \xrightarrow{\ \varepsilon\ } & R[0] & \longrightarrow & C & \longrightarrow & R[1] \\
\| & & \downarrow{\scriptstyle 1+\varepsilon} & & \| & & \| \\
R[0] & \xrightarrow{\ \varepsilon\ } & R[0] & \longrightarrow & C & \longrightarrow & R[1]
\end{array}
$$

where $C = \mathrm{cone}\big(R[0] \xrightarrow{\varepsilon} R[0]\big)$, compare [10, §2]. $\qquad\square$

Corollary 4.2. *If k is a finite field and $R = k[\varepsilon]/(\varepsilon^2)$ then the groups $K_1(R)$ and $K_1\big(\mathbf{K}^{\mathrm{b}}(R\text{-proj})\big)$ are not isomorphic.*

Proof. Lemma 4.1 shows that in this case the canonical map $K_1(R) \to K_1\big(\mathbf{K}^{\mathrm{b}}(R\text{-proj})\big)$ is surjective but not injective. Since $K_1(R) \cong R^\times$ is finite, it follows that $K_1(R)$ and $K_1\big(\mathbf{K}^{\mathrm{b}}(R\text{-proj})\big)$ are not isomorphic. $\qquad\square$

Corollary 4.3. *If k is a finite field and $R = k[\varepsilon]/(\varepsilon^2)$ then the groups $K_1(R) = K_1(R\text{-proj})$ and $K_1\big(\mathbf{D}^{\mathrm{b}}(R\text{-proj})\big)$ are not isomorphic.*

Proof. Recall that the derived category $\mathbf{D}^{\mathrm{b}}(R\text{-proj})$ is obtained from $\mathbf{K}^{\mathrm{b}}(R\text{-proj})$ by inverting all morphisms whose cone is acyclic (in the sense of [7, p. 389]). But it is easy to see that the cone of a morphism $a : A \to B$ in $\mathbf{K}^{\mathrm{b}}(R\text{-proj})$ is acyclic if and only if a is an isomorphism. Hence $\mathbf{D}^{\mathrm{b}}(R\text{-proj}) = \mathbf{K}^{\mathrm{b}}(R\text{-proj})$ and therefore $K_1\big(\mathbf{D}^{\mathrm{b}}(R\text{-proj})\big) = K_1\big(\mathbf{K}^{\mathrm{b}}(R\text{-proj})\big)$. Thus Corollary 4.3 follows from Corollary 4.2. $\qquad\square$

Corollary 4.4. *If k is a finite field and $R = k[\varepsilon]/(\varepsilon^2)$ then the groups $K_1(R)$ and $K_1\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$ are not isomorphic.*

Proof. The canonical functor $\mathbf{K}^{\mathrm{b}}(R\text{-proj}) \to \mathbf{D}^{\mathrm{perf}}(R)$ is an equivalence of triangulated categories. Hence $K_1\big(\mathbf{K}^{\mathrm{b}}(R\text{-proj})\big) \cong K_1\big(\mathbf{D}^{\mathrm{perf}}(R)\big)$ by [1, Corollary 4.11]. Thus Corollary 4.4 follows from Corollary 4.2. $\qquad\square$

5. HOMOTOPY FIBRES OF MONOIDAL FUNCTORS

In this section we summarize the necessary facts about the homotopy fibre of a monoidal functor of Picard categories. These constructions and results are well-known (cf. [4], [2, §5]), the only difference in our presentation here is the absence of a fixed unit object.

Let $M = (M, c) : \mathcal{P} \to \mathcal{P}'$ be a monoidal functor of Picard categories. The homotopy fibre of M is the Picard category $\mathcal{F}(M)$ defined as follows. Objects of $\mathcal{F}(M)$ are pairs (X, δ) where X is an object of $\mathcal{P}$ and $\delta : M(X) \to M(X) \otimes M(X)$ is a unit structure on $M(X)$. A morphism $(X, \delta) \to (Y, \varepsilon)$ in $\mathcal{F}(M)$ is a morphism $\alpha : X \to Y$ in $\mathcal{P}$ such that $\varepsilon \circ M(\alpha) = (M(\alpha) \otimes M(\alpha)) \circ \delta$. The composition of morphisms in $\mathcal{F}(M)$ is given by the composition of morphisms in $\mathcal{P}$.

The $\otimes$-product of (X, δ) and (Y, ε) is $(X, \delta) \otimes (Y, \varepsilon) = (X \otimes Y, \gamma)$ where γ is induced by the isomorphism $M(X \otimes Y) \xrightarrow{c_{X,Y}} M(X) \otimes M(Y)$ and the product unit structure $(M(X), \delta) \otimes (M(Y), \varepsilon)$, i.e. γ is the composite isomorphism $M(X \otimes Y) \cong M(X) \otimes M(Y) \xrightarrow{\delta \otimes \varepsilon} \big(M(X) \otimes M(X)\big) \otimes \big(M(Y) \otimes M(Y)\big) \cong \big(M(X) \otimes M(Y)\big) \otimes \big(M(X) \otimes M(Y)\big) \cong M(X \otimes Y) \otimes M(X \otimes Y)$. The $\otimes$-product of two morphisms in $\mathcal{F}(M)$ is simply the $\otimes$-product of these morphisms in $\mathcal{P}$.

The AC-structure on $\mathcal{F}(M)$ is induced by the AC-tensor structure on $\mathcal{P}$, i.e. $\psi_{(X,\delta),(Y,\varepsilon)} : (X, \delta) \otimes (Y, \varepsilon) \cong (Y, \varepsilon) \otimes (X, \delta)$ is given by $\psi_{X,Y} : X \otimes Y \cong Y \otimes X$, and $\varphi_{(X,\delta),(Y,\varepsilon),(Z,\gamma)} = \varphi_{X,Y,Z}$.

There exists an obvious monoidal functor $J : \mathcal{F}(M) \to \mathcal{P}$ which sends an object (X, δ) to X and a morphism α to α. Applying π_i to this functor gives homomorphisms $\pi_i(J) : \pi_i(\mathcal{F}(M)) \to \pi_i(\mathcal{P})$ for $i = 0, 1$. Applying π_i to the functor $M : \mathcal{P} \to \mathcal{P}'$ gives homomorphisms $\pi_i(M) : \pi_i(\mathcal{P}) \to \pi_i(\mathcal{P}')$ for $i = 0, 1$. Finally there is a homomorphism $\partial^1 : \pi_1(\mathcal{P}') \to \pi_0(\mathcal{F}(M))$ which sends $\alpha \in \pi_1(\mathcal{P}')$ to the isomorphism class of (U, δ) where $\gamma : U \cong U \otimes U$ is any unit in $\mathcal{P}$ and δ is the composite isomorphism $M(U) \xrightarrow{\alpha} M(U) \xrightarrow{M(\gamma)} M(U \otimes U) \xrightarrow{c_{U,U}} M(U) \otimes M(U)$.

The following lemma is well known and easy to verify.

Lemma 5.1. *There is an exact sequence of homotopy groups*

$$0 \to \pi_1(\mathcal{F}(M)) \xrightarrow{\pi_1(J)} \pi_1(\mathcal{P}) \xrightarrow{\pi_1(M)} \pi_1(\mathcal{P}') \xrightarrow{\partial^1} \pi_0(\mathcal{F}(M)) \xrightarrow{\pi_0(J)} \pi_0(\mathcal{P}) \xrightarrow{\pi_0(M)} \pi_0(\mathcal{P}').$$

The homotopy fibre and the associated exact sequence of homotopy groups are functorial in the following sense. Given a diagram

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{M} & \mathcal{P}' \\
\downarrow{\scriptstyle A} & & \downarrow{\scriptstyle B} \\
\mathcal{Q} & \xrightarrow{N} & \mathcal{Q}'
\end{array}
$$

of Picard categories and monoidal functors, and an isomorphism $\kappa : B \circ M \to N \circ A$ of monoidal functors, we obtain a monoidal functor $\mathcal{F}(M) \to \mathcal{F}(N)$ of the homotopy fibres, which sends an object (X, δ) in $\mathcal{F}(M)$ to the object $(A(X), \delta')$ in $\mathcal{F}(N)$, where δ' is the composite $N(A(X)) \xrightarrow{\kappa_X^{-1}} B(M(X)) \xrightarrow{B(\delta)} B(M(X) \otimes M(X)) \cong B(M(X)) \otimes B(M(X)) \xrightarrow{\kappa_X \otimes \kappa_X} N(A(X)) \otimes N(A(X))$, and a morphism $\alpha : (X, \delta) \to (Y, \varepsilon)$ to the morphism $A(\alpha)$. It is not difficult to verify that the induced homomorphisms $\pi_i(\mathcal{F}(M)) \to \pi_i(\mathcal{F}(N))$ for $i = 0, 1$ make the diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \pi_1(\mathcal{F}(M)) & \longrightarrow & \pi_1(\mathcal{P}) & \longrightarrow & \pi_1(\mathcal{P}') & \longrightarrow & \pi_0(\mathcal{F}(M)) & \longrightarrow & \pi_0(\mathcal{P}) & \longrightarrow & \pi_0(\mathcal{P}') \\
& & \downarrow & & \downarrow{\scriptstyle \pi_1(A)} & & \downarrow{\scriptstyle \pi_1(B)} & & \downarrow & & \downarrow{\scriptstyle \pi_0(A)} & & \downarrow{\scriptstyle \pi_0(B)} \\
0 & \longrightarrow & \pi_1(\mathcal{F}(N)) & \longrightarrow & \pi_1(\mathcal{Q}) & \longrightarrow & \pi_1(\mathcal{Q}') & \longrightarrow & \pi_0(\mathcal{F}(N)) & \longrightarrow & \pi_0(\mathcal{Q}) & \longrightarrow & \pi_0(\mathcal{Q}')
\end{array}
$$

commutative.

Remark 5.2. Giving the structure of a unit on $M(X)$ is equivalent to giving an isomorphism $M(X) \to 1_{\mathcal{P}'}$ where $1_{\mathcal{P}'} \to 1_{\mathcal{P}'} \otimes 1_{\mathcal{P}'}$ is a fixed unit in $\mathcal{P}'$. Therefore the homotopy fibre defined above agrees with the fibre product considered in [4].

6. Euler characteristics via triangulated categories

In this section we describe the construction of Euler characteristics in a relative algebraic K_0-group. The construction here is essentially the same as in [2], except that we work with determinant functors on triangulated categories of perfect complexes instead of determinant functors on exact categories of bounded complexes.

Let $R \to S$ be a homomorphism of rings such that S is flat as right R-module. Furthermore we assume that S is regular. Let $K_0(R, S)$ be the relative algebraic K-group which is defined in terms of generators and relations in [9, p. 215].

We fix universal determinant functors $g_R : \mathbf{D}^{\mathrm{perf}}(R) \to \mathcal{W}(R)$ and $g_S : \mathbf{D}^{\mathrm{perf}}(S) \to \mathcal{W}(S)$. If $F : \mathbf{D}^{\mathrm{perf}}(R) \to \mathbf{D}^{\mathrm{perf}}(S)$ denotes the functor given by the scalar extension $P \mapsto S \otimes_R P$ then there exists a monoidal functor $N : \mathcal{W}(R) \to \mathcal{W}(S)$ such that the determinant functors $N \circ g_R$ and $g_S \circ F$ are isomorphic. We fix such a functor N and isomorphism $\mu : N \circ g_R \cong g_S \circ F$.

Lemma 6.1. *Let* $N : \mathcal{W}(R) \to \mathcal{W}(S)$ *and* $\mu : N \circ g_R \cong g_S \circ F$ *be as above, and let* $\mathcal{F}(N)$ *denote the homotopy fibre of* N. *Then there exists an isomorphism* $\eta : K_0(R, S) \cong \pi_0(\mathcal{F}(N))$ *(depending on* μ*).*

We will prove Lemma 6.1 below. The isomorphism $\eta : K_0(R, S) \cong \pi_0(\mathcal{F}(N))$ allows us to construct invariants in $K_0(R, S)$ using the determinant functors g_R and g_S. If C is a perfect complex of R-modules and ε is a unit structure on $N(g_R(C))$, then $(g_R(C), \varepsilon)$ is an object in $\mathcal{F}(N)$ and therefore has a class in $\pi_0(\mathcal{F}(N)) \cong K_0(R, S)$. The isomorphism $\mu_C : N(g_R(C)) \cong g_S(S \otimes_R C)$ and the fact that S is regular allow us to construct a unit structure on $N(g_R(C))$ from certain information about the cohomology of $S \otimes_R C$. The relevant properties of the determinant of the cohomology are summarized in the following lemma (which is proved later in this section).

Lemma 6.2. *Let* S *be a regular ring and* $g : \mathbf{D}^{\mathrm{perf}}(S) \to \mathcal{P}$ *a determinant functor. Let* P *be a perfect complex of* S-modules.

   (1) *Let* $H(P)$ *denote the cohomology of* P *considered as a complex with zero differentials. Then* $H(P)$ *is a bounded complex of finitely generated* S-modules *(and so in particular it lies in* $\mathbf{D}^{\mathrm{perf}}(S)$*), and there exists a canonical isomorphism*

$$g(P) \cong g(H(P)).$$

   (2) *Let* $H^{\mathrm{ev}}(P)$ *resp.* $H^{\mathrm{od}}(P)$ *denote the direct sum of the even resp. odd cohomology of* P. *Then there exists a canonical isomorphism*

$$g(H(P)) \cong g(H^{\mathrm{ev}}(P)[0]) \otimes g(H^{\mathrm{od}}(P)[1]).$$

   (3) *There exists a canonical unit structure on* $g(H^{\mathrm{od}}(P)[0]) \otimes g(H^{\mathrm{od}}(P)[1])$.

Using Lemmas 6.1 and 6.2 we can now define Euler characteristics in $K_0(R, S)$. To simplify the notation we will write C_S for $S \otimes_R C$ if C is a complex of R-modules.

Definition 6.3. Let C be a perfect complex of R-modules and $t : H^{\mathrm{ev}}(C_S) \xrightarrow{\cong} H^{\mathrm{od}}(C_S)$ an isomorphism of S-modules. We define a unit structure ε on $N(g_R(C))$

via the composite isomorphism

$$N(g_R(C)) \xrightarrow{\mu_C} g_S(C_S)$$
$$\xrightarrow{\cong} g_S(H(C_S))$$
$$\xrightarrow{\cong} g_S(H^{\mathrm{ev}}(C_S)[0]) \otimes g_S(H^{\mathrm{od}}(C_S)[1])$$
$$\xrightarrow{g_S(t)\otimes\mathrm{id}} g_S(H^{\mathrm{od}}(C_S)[0]) \otimes g_S(H^{\mathrm{od}}(C_S)[1])$$

and the canonical unit structure on $g_S(H^{\mathrm{od}}(C_S)[0]) \otimes g_S(H^{\mathrm{od}}(C_S)[1])$. Then the Euler characteristic $\chi^{\mathrm{tri}}(C,t) \in K_0(R,S)$ of the pair (C,t) is defined to be the isomorphism class of the object $(g_R(C),\varepsilon)$ in $\pi_0(\mathcal{F}(N)) \xrightarrow{\eta^{-1}} K_0(R,S)$.

Lemma 6.4. *The definition of $\chi^{\mathrm{tri}}(C,t) \in K_0(R,S)$ is independent of all choices.*

Lemma 6.5. *Let $\chi(C,t)$ be the Euler characteristic defined in [2, Definition 5.5]. Then $\chi^{\mathrm{tri}}(C,t) = \chi(C,t)$.*

We remark that Lemma 6.4 follows immediately from Lemma 6.5 and the independence of $\chi(C,t)$ of all choices. However the latter independence was only stated without proof in [2, Remark 5.3], and we will therefore include a complete proof of Lemma 6.4 below.

Proof of Lemma 6.1. This proof is similar to [2, Lemma 5.1]. We define a homomorphism $\eta : K_0(R,S) \to \pi_0(\mathcal{F}(N))$ by sending a generator (P,a,Q) to the isomorphism class of $(g_R(P[0]) \otimes g_R(Q[0])^{-1}, \delta)$, where the unit structure δ on $N(g_R(P[0]) \otimes g_R(Q[0])^{-1})$ is obtained via the composite isomorphism

$$N(g_R(P[0]) \otimes g_R(Q[0])^{-1}) \xrightarrow{\cong} N(g_R(P[0])) \otimes N(g_R(Q[0]))^{-1}$$
$$\xrightarrow{\mu} g_S(S \otimes_R P[0]) \otimes g_S(S \otimes_R Q[0])^{-1}$$
$$\xrightarrow{g_S(a[0])\otimes\mathrm{id}} g_S(S \otimes_R Q[0]) \otimes g_S(S \otimes_R Q[0])^{-1}$$

(where μ denotes the isomorphism induced by $\mu_{P[0]} : N(g_R(P[0])) \cong g_S(S \otimes_R P[0])$ and $\mu_{Q[0]} : N(g_R(Q[0])) \cong g_S(S \otimes_R Q[0])$) and the canonical unit structure on $g_S(S \otimes_R Q[0]) \otimes g_S(S \otimes_R Q[0])^{-1}$. We now have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
K_1(R) & \longrightarrow & K_1(S) & \longrightarrow & K_0(R,S) & \longrightarrow & K_0(R) & \longrightarrow & K_0(S) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\eta} & & \downarrow & & \downarrow \\
K_1(\mathbf{D}^{\mathrm{perf}}(R)) & \longrightarrow & K_1(\mathbf{D}^{\mathrm{perf}}(S)) & \longrightarrow & \pi_0(\mathcal{F}(N)) & \longrightarrow & K_0(\mathbf{D}^{\mathrm{perf}}(R)) & \longrightarrow & K_0(\mathbf{D}^{\mathrm{perf}}(S))
\end{array}
$$

in which the unlabeled vertical maps are the canonical homomorphisms from Theorem 1.1. Since the first vertical map is surjective and the second, fourth and fifth vertical maps are isomorphisms, the 5-Lemma implies that η is an isomorphism. $\square$

Proof of Lemma 6.2. The canonical functor $\mathbf{C}^{\mathrm{b}}(S\text{-mod}) \to \mathbf{D}^{\mathrm{perf}}(S)$ induces a functor

$$\det\left(\mathbf{D}^{\mathrm{perf}}(S), \mathcal{P}\right) \longrightarrow \det\left((\mathbf{C}^{\mathrm{b}}(S\text{-mod}), \mathrm{qis}), \mathcal{P}\right)$$

(this is the composite of the functor $\det\left(\mathbf{D}^{\mathrm{perf}}(S), \mathcal{P}\right) \to \det\left((\mathbf{C}^{\mathrm{perf}}(S), \mathrm{qis}), \mathcal{P}\right)$ from Lemma 3.1 and the functor $\det\left((\mathbf{C}^{\mathrm{perf}}(S), \mathrm{qis}), \mathcal{P}\right) \to \det\left((\mathbf{C}^{\mathrm{b}}(S\text{-mod}), \mathrm{qis}), \mathcal{P}\right)$

induced by $\mathbf{C}^{\mathrm{b}}(S\text{-mod}) \to \mathbf{C}^{\mathrm{perf}}(S)$). Let $f : (\mathbf{C}^{\mathrm{b}}(S\text{-mod}), \mathrm{qis}) \to \mathcal{P}$ denote the image of $g : \mathbf{D}^{\mathrm{perf}}(S) \to \mathcal{P}$ under this functor.
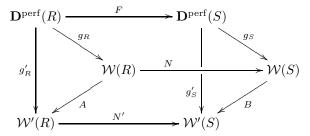
(1) Choose a complex U in $\mathbf{C}^{\mathrm{b}}(S\text{-mod})$ together with a quasi-isomorphism $a : U \to P$. Then we get an isomorphism $H(a) : H(U) \to H(P)$, and it is clear that $H(U)$ (and hence $H(P)$) lies in $\mathbf{C}^{\mathrm{b}}(S\text{-mod})$. Now [2, Proposition 3.1] shows that there exists a canonical isomorphism $f(U) \xrightarrow{\cong} f(H(U))$. Hence we obtain an isomorphism $g(P) \xrightarrow{\cong} g(H(P))$ as the composite

$$g(P) \xrightarrow{g(a)^{-1}} g(U) = f(U) \xrightarrow{\cong} f(H(U)) = g(H(U)) \xrightarrow{g(H(a))} g(H(P)).$$

One easily checks that this composite isomorphism is independent of the choice of U and a.

(2) This follows from the canonical isomorphism $f(H(P)) \cong f(H^{\mathrm{ev}}(P)[0]) \otimes f(H^{\mathrm{od}}(P)[1])$, see [2, Proposition 4.4].

(3) This follows from the canonical unit structure on $f(H^{\mathrm{od}}(P)[0]) \otimes f(H^{\mathrm{od}}(P)[1])$, see [2, Lemma 2.3]. □

Proof of Lemma 6.4. Suppose that $g'_R : \mathbf{D}^{\mathrm{perf}}(R) \to \mathcal{W}'(R)$ and $g'_S : \mathbf{D}^{\mathrm{perf}}(S) \to \mathcal{W}'(S)$ are also universal determinant functor, $N' : \mathcal{W}'(R) \to \mathcal{W}'(S)$ is a monoidal functor, and $\mu' : N' \circ g'_R \cong g'_S \circ F$ is an isomorphism. Then we can choose a monoidal functor $A : \mathcal{W}(R) \to \mathcal{W}'(R)$ together with an isomorphism $A \circ g_R \cong g'_R$, and a monoidal functor $B : \mathcal{W}(S) \to \mathcal{W}'(S)$ together with an isomorphism $B \circ g_S \cong g'_S$. So we have a diagram of triangulated/Picard categories and exact/determinant/monoidal functors as follows.

From μ, μ' and the isomorphism of the triangles we obtain an isomorphism of determinant functors

$$\mathbf{D}^{\mathrm{perf}}(R) \xrightarrow{g_R} \mathcal{W}(R) \xrightarrow{N} \mathcal{W}(S) \xrightarrow{B} \mathcal{W}'(S),$$

$$\mathbf{D}^{\mathrm{perf}}(R) \xrightarrow{g_R} \mathcal{W}(R) \xrightarrow{A} \mathcal{W}'(R) \xrightarrow{N'} \mathcal{W}'(S),$$

and since g_R is universal this isomorphism of determinant functors comes from an isomorphism of monoidal functors $B \circ N \cong N' \circ A$. Therefore by the results in §5 we get an induced functor $Z : \mathcal{F}(N) \to \mathcal{F}(N')$ and thus a homomorphism $\pi_0(Z) : \pi_0(\mathcal{F}(N)) \to \pi_0(\mathcal{F}(N'))$. One easily sees that $\pi_0(Z)$ is in fact an isomorphism.

We now write $\eta_{N,\mu} : K_0(R, S) \to \pi_0(\mathcal{F}(N))$ for the isomorphism from Lemma 6.1 associated to N and μ, and $\eta_{N',\mu'} : K_0(R, S) \to \pi_0(\mathcal{F}(N'))$ for the isomorphism associated to N' and μ'. It is not difficult to verify that

$$(9) \qquad\qquad\qquad \eta_{N',\mu'} = \pi_0(Z) \circ \eta_{N,\mu}.$$

On the other hand, let $(g_R(C), \varepsilon)$ be the object in $\mathcal{F}(N)$ which occurs in the construction of the Euler characteristic with respect to N and μ, and let $(g'_R(C), \varepsilon')$

be the object in $\mathcal{F}(N')$ constructed with respect to N' and μ'. It is then easy to see that $Z((g_R(C), \varepsilon)) \cong (g'_R(C), \varepsilon')$. Hence $\pi_0(Z)$ sends the isomorphism class of $(g_R(C), \varepsilon)$ to the isomorphism class of $(g'_R(C), \varepsilon')$. Together with (9) this implies the independence of $\chi^{\mathrm{tri}}(C, t)$. $\qquad\qquad\square$

Proof of Lemma 6.5. It is easy to check that if $a : U \to C$ is a quasi-isomorphism and $t' : H^{\mathrm{ev}}(U_S) \to H^{\mathrm{od}}(U_S)$ is the composite isomorphism
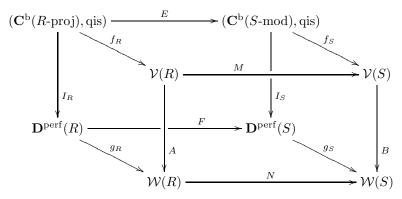
$$H^{\mathrm{ev}}(U_S) \xrightarrow{H^{\mathrm{ev}}(a_S)} H^{\mathrm{ev}}(C_S) \xrightarrow{t} H^{\mathrm{od}}(C_S) \xrightarrow{H^{\mathrm{od}}(a_S)^{-1}} H^{\mathrm{od}}(U_S),$$

then $\chi^{\mathrm{tri}}(C, t) = \chi^{\mathrm{tri}}(U, t')$ and $\chi(C, t) = \chi(U, t')$. Therefore we can assume from now on that C is a bounded complex of finitely generated projective R-modules.

Let $f_R : (\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}) \to \mathcal{V}(R)$ and $f_S : (\mathbf{C}^{\mathrm{b}}(S\text{-mod}), \mathrm{qis}) \to \mathcal{V}(S)$ be universal determinant functors. Let $E : \mathbf{C}^{\mathrm{b}}(R\text{-proj}) \to \mathbf{C}^{\mathrm{b}}(S\text{-mod})$ be the exact functor given by $P \mapsto S \otimes_R P$, and let $M : \mathcal{V}(R) \to \mathcal{V}(S)$ be a monoidal functor such that there exists an isomorphism of determinant functors $\lambda : M \circ f_R \cong f_S \circ E$. Then by [2, Lemma 5.1] there exists an isomorphism $\xi : K_0(R, S) \cong \pi_0(\mathcal{F}(M))$ (depending on λ).

We write I_R for the canonical functor $\mathbf{C}^{\mathrm{b}}(R\text{-proj}) \to \mathbf{D}^{\mathrm{perf}}(R)$. Then there exists a monoidal functor $A : \mathcal{V}(R) \to \mathcal{W}(R)$ such that the determinant functors $g_R \circ I_R$ and $A \circ f_R$ are isomorphic. We fix such a functor A and isomorphism of determinant functors. Similarly, if $I_S : \mathbf{C}^{\mathrm{b}}(S\text{-mod}) \to \mathbf{D}^{\mathrm{perf}}(S)$ is the canonical functor, we can fix a monoidal functor $B : \mathcal{V}(S) \to \mathcal{W}(S)$ and isomorphism $g_S \circ I_S \cong B \circ f_S$.

We now have the following diagram of exact/triangulated/Picard categories and exact/determinant/monoidal functors.

The square at the back is commutative. The squares on the left, right, top and bottom are commutative up to fixed isomorphisms of determinant functors. Hence we obtain an isomorphism of determinant functors

$$(\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}) \xrightarrow{f_R} \mathcal{V}(R) \xrightarrow{M} \mathcal{V}(S) \xrightarrow{B} \mathcal{W}(S),$$

$$(\mathbf{C}^{\mathrm{b}}(R\text{-proj}), \mathrm{qis}) \xrightarrow{f_R} \mathcal{V}(R) \xrightarrow{A} \mathcal{W}(R) \xrightarrow{N} \mathcal{W}(S),$$

and thus (since f_R is universal) also an isomorphism of monoidal functors $B \circ M \cong N \circ A$. As shown in §5 this implies that there is a monoidal functor $Z : \mathcal{F}(M) \to \mathcal{F}(N)$. It follows easily from the proofs of Lemma 6.1 and [2, Lemma 5.1] that

$$(10) \qquad\qquad\qquad \eta = \pi_0(Z) \circ \xi.$$

Now recall that the Euler characteristic $\chi(C,t) \in K_0(R,S)$ is defined to be the isomorphism class of $(f_R(C), \delta)$ in $\pi_0(\mathcal{F}(M)) \xrightarrow{\xi^{-1}} K_0(R,S)$, where δ is the unit structure on $M(f_R(C))$ coming from the isomorphism

$$M(f_R(C)) \xrightarrow{\lambda_C} f_S(C_S) \xrightarrow{\cong} f_S(H(C_S)) \xrightarrow{\cong} f_S(H^{\mathrm{ev}}(C_S)[0]) \otimes f_S(H^{\mathrm{od}}(C_S)[1])$$

$$\xrightarrow{f_S(t)\otimes\mathrm{id}} f_S(H^{\mathrm{od}}(C_S)[0]) \otimes f_S(H^{\mathrm{od}}(C_S)[1])$$

and the canonical unit structure on $f_S(H^{\mathrm{od}}(C_S)[0]) \otimes f_S(H^{\mathrm{od}}(C_S)[1])$.

It is straightforward to check that $Z((f_R(C), \delta)) \cong (g_R(C), \varepsilon)$ where $(g_R(C), \varepsilon)$ is the object in $\mathcal{F}(N)$ which is used in the definition of $\chi^{\mathrm{tri}}(C,t)$. Because of (10) this implies that $\chi^{\mathrm{tri}}(C,t) = \chi(C,t)$ as required. $\qquad\square$

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DEPARTMENT OF MATHEMATICS, KING'S COLLEGE LONDON, STRAND, LONDON WC2R 2LS, UNITED KINGDOM

E-mail address: `manuel.breuning@kcl.ac.uk`